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On Markov's Inequality on R for the Hermite Weight

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The best constant in Markov's inequality on R for the Hermite weight is characterized in terms of the weighted Chebyshev polynomial. © 1993 Academic Press, Inc.

1. INTRODUCTION

In recent years there has been considerable research activities in the area of weighted polynomial approximation on R. As an important tool, Markov type inequalities have been established for various weights (cf., e.g., [3, 4, 8]). Markov type inequalities are of the form

$$\|(wp_n)'\| \leqslant K_w(n)\, \|wp_n\|,$$

where w is a weight on R, $K_w(n)$, a quantity depending on w and n, $\|\cdot\|$, the sup norm on R, and $p_n \in \mathscr{P}_n$, the set of real polynomials of degree at most n. In several interesting cases, the estimates for $K_w(n)$ as $n \to \infty$ have been established (cf. [3, 4, 8]). Obviously, the optimal choice of $K_w(n)$ will be

$$C_w(n) := \sup_{\substack{p_n \in \mathscr{P}_n \\ p_n \not\equiv 0}} \frac{\|(wp_n)'\|}{\|wp_n\|}.$$

In this form, $C_w(n)$ is the extreme value of the following extremal problem:

$$\text{(P)} \quad \begin{cases} \text{maximize } \|(wp_n)'\| \\ \text{subject to} \\ p_n \in \mathscr{P}_n \quad \text{and} \quad \|wp_n\| \leqslant 1. \end{cases}$$

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If $w(x) = \chi_{[-1,1]}(x)$ (the characteristic function of $[-1, 1]$), then by the classical Markov inequality,

$$C_{\chi_{[-1,1]}}(n) = \frac{\|T_n'\|_{[-1,1]}}{\|T_n\|_{[-1,1]}} = n^2,$$

where $T_n(x) = x^n + \cdots = 2^{1-n} \cos n \arccos x$ (the nth Chebyshev polynomial) and $\|\cdot\|_{[-1,1]}$ is the sup norm on $[-1, 1]$. From this, one would conjecture that under "suitable" conditions on w,

$$C_w(n) = \frac{\|(w(x)\, T_n(x, w))'\|}{\|w(x)\, T_n(x, w)\|}, \tag{1}$$

where $T_n(x, w) = x^n + \cdots \in \mathscr{P}_n$ is the *weighted Chebyshev polynomial* of degree n, i.e., $T_n(x, w)$ satisfies

$$\|w(x)\, T_n(x, w)\| = \inf_{p \in \mathscr{P}_{n-1}} \|w(x)(x^n + p(x))\|.$$

It is known (cf. [7]) that $T_n(\cdot, w)$ can be characterized by the maximal equioscillation property.

The purpose of this paper is to show that (1) is true for the important case when $w(x) = w_2(x) := e^{-x^2}$, the Hermite weight. This problem is partially resolved in [6]. Mohapatra *et al.* showed that $\pm \hat{T}_n := T_n(\cdot, w_2)/\|w_2 T(\cdot, w_2)\|$ and $\pm \hat{T}_{n-1}$ are the only candidates for the solution of problem (P). However, the task that eliminates $\pm \hat{T}_{n-1}$ as a possible solution is not trivial. By means of a representation theorem in [2] and analysis used in [9] for extremal problems, we have been able to show that the solution of the problem (P) when $w(x) = w_2(x)$ is $\pm \hat{T}_n$. More precisely, we prove the following:

THEOREM 1. *With the notation mentioned above,*

$$\max_{\substack{p_n \in \mathscr{P}_n \\ p_n \not\equiv 0}} \frac{\|(w_2 p_n)'\|}{\|w_2 p_n\|} = \frac{\|(w_2 \hat{T}_n)'\|}{\|w_2 \hat{T}_n\|}. \tag{2}$$

It is hoped that the result of this paper will lead to deeper research to establish the optimal value $C_w(n)$ in more general settings.

The paper is organized as follows: In Section 2, we prove some preliminary results for general weights; In Section 3, we concentrate on the case of the Hermite weight and prove Theorem 1; In Section 4, we give consequences of Theorem 1 and related remarks.

2. Preliminary Results

Let the weight $w: R \to (0, \infty)$ be continuously differentiable, $w(x)|x|^k \to 0$ as $|x| \to \infty$ ($k = 0, 1, 2, ...$), and w'/w be continuous and decreasing.

Our proof of Theorem 1 requires a number of lemmas. Before we mention the specific results, we give a sketch of the ideas involved in the proof (cf. [9]).

We need to show that $\pm \hat{T}_n$ is the only solution of problem (P). To do so, we first consider a pointwise version of the problem (P) which can be stated (in a form convenient to our later discussion) as (for $y \in R$)

$$(\mathrm{P}_y) \qquad \begin{cases} \text{minimize } -(wp_n)'(y) \\ \text{subject to} \\ p_n \in \mathscr{P}_n \qquad \text{and} \qquad \|wp_n\| \leqslant 1. \end{cases}$$

By a standard compactness argument, the existence of solution to (P_y) can be easily established. Let $N_n(y)$ be the negative of the extremal value, i.e.,

$$N_n(y) := - \min_{\substack{p_n \in \mathscr{P}_n \\ \|wp_n\| \leqslant 1}} [-(wp_n)'(y)]. \tag{3}$$

(We will see that $N_n(y) > 0$. See the remark after the proof of Lemma 3 in Section 2.) After proving the uniqueness of the solution of the problem (P_y), we then determine a closed set $I \subset R$ such that

$$N_n(y) = |(w\hat{T}_n)'(y)|, \qquad y \in I,$$

and

$$N_n(y) > |(w\hat{T}_n)'(y)|, \qquad y \notin I.$$

Finally, when $w = w_2$, we show that

$$\sup_{y \notin I} N_n(y) < \sup_{y \in I} N_n(y) = \max_{y \in I} N_n(y).$$

Thus, for $w = w_2$, $\max_{y \in R} N_n(y) = C_w(n)$ is attained only by $\pm \hat{T}_n(\cdot, w_2)$.

The following result established in [6] is needed in our proof.

Lemma 2 (cf. [6, Lemma 5 and Its Proof]). *Suppose $p \in \mathscr{P}_n$ has n distinct real zeros. Then there are exactly $(n+1)$ distinct real numbers where $(wp)'$ vanishes. Furthermore, the $(n+1)$ zeros of $(wp)'$ and the n zeros of p are interlacing.*

We now consider problem (P_y). The Corollary on p. 84 in [2] yields that $Q_n = Q_n(\cdot, y) \in \mathscr{P}_n$ is a solution of (P_y) if and only if there exist

$\lambda_j = \lambda_j(y) \neq 0$ and $\tau_j = \tau_j(y)$, $j = 1, 2, ..., r$, for some $r = r(y)$, $0 \leqslant r \leqslant n+1$, and

$$\tau_1 < \tau_2 < \cdots < \tau_r$$

such that

$$\begin{aligned} (wp_n)'(y) &= \sum_{j=1}^{r} \lambda_j (wp_n)(\tau_j), && \text{for all} \quad p_n \in \mathscr{P}_n, \\ \operatorname{sgn} \lambda_j &= \operatorname{sgn}(wQ_n)(\tau_j), && \text{and} \\ |(wQ_n)(\tau_j)| &= \|wQ_n\| = 1, && j = 1, 2, ..., r. \end{aligned} \tag{4}$$

Since Theorem 1 can be established by direct computation for $n = 1, 2$, as indicated in [6, Remarks], we will assume, from now on, that $n \geqslant 3$.

LEMMA 3. *Assume Q_n is a solution of (P_y), then we have $r \geqslant n$ in* (4) *and that $(wQ_n)'$ has exactly $(n+1)$ distinct zeros.*

Proof. We show $r \geqslant n$ by contradiction. Assume $r \leqslant n-1$. Taking $p_n(x) = \prod_{j=1}^{r} (x - \tau_j) \in \mathscr{P}_{n-1}$ in (4) gives $(wp_n)'(y) = 0$. But using $xp_n(x)$ instead of $p_n(x)$ in (4) yields $(wp_n)'(y) \cdot y + (wp_n)(y) = 0$, so $(wp_n)(y) = 0$ or $p_n(y) = 0$, thus $y = \tau_j$ for some j and so $(wQ_n)'(y) = 0$. Then, for any $q_n \in \mathscr{P}_n$ with $\|wq_n\| \leqslant 1$,

$$-(wq_n)'(y) \geqslant -(wQ_n)'(y) = 0,$$

by the extremality of Q_n. This would imply that both $w(y) = 0$ and $w'(y) = 0$. But w is a positive weight. Hence we get a contradiction.

Now, note that Q_n itself must have at least $(n-1)$ sign changes at τ_j's. In fact, if the sequence $Q_n(\tau_1)$, $Q_n(\tau_2)$, ..., $Q_n(\tau_r)$ changes sign less than $(n-1)$ times, then we can find a polynomial, say $q_{n-2} \in \mathscr{P}_{n-2}$, having the same sign as Q_n at τ_j, $j = 1, 2, ..., r$. Taking $p_n(x) = (x-y)^2 q_{n-2}(x)$ in (4) gives

$$0 = \sum_{j=1}^{r} |\lambda_j| \operatorname{sgn} Q_n(\tau_j) \cdot w(\tau_j)(\tau_j - y)^2 q_{n-2}(\tau_j) > 0,$$

a contradiction. So Q_n has at least $(n-1)$ sign changes, and thus has n real distinct zeros. (Recall that Q_n is a real polynomial). From this, by Lemma 2, $(wQ_n)'$ must have exactly $(n+1)$ distinct zeros. ∎

Remark. From the proof of Lemma 3 (the first paragraph), we see that generally it is true that $N_n(y) \neq 0$. But it is immediate from the definition of $N_n(y)$ that $N_n(y) \geqslant 0$, so $N_n(y) > 0$ for all $y \in R$.

LEMMA 4. *There exists a unique solution to problem* (P_y).

Proof. Since the existence of the solution is mentioned before, we need only to show the uniqueness. Let Q_n be a solution. Then we have (4). Assume K_n is another solution of (P_y). Then

$$(wK_n)'(y) = (wQ_n)'(y) \tag{5}$$

and

$$\|wK_n\| = 1. \tag{6}$$

Now by (4)

$$(wK_n)'(y) = \sum_{j=1}^{r} \lambda_j (wK_n)(\tau_j)$$

and

$$(wQ_n)'(y) = \sum_{j=1}^{r} \lambda_j (wQ_n)(\tau_j) = \sum_{j=1}^{r} |\lambda_j|,$$

thus, Eq. (5) yields

$$\sum_{j=1}^{r} \lambda_j (wK_n)(\tau_j) = \sum_{j=1}^{r} |\lambda_j|.$$

In view of (6), this implies

$$(wK_n)(\tau_j) = \operatorname{sgn} \lambda_j, \qquad \text{and} \qquad (wK_n)'(\tau_j) = 0, \qquad j = 1, 2, ..., r.$$

Then it follows that

$$w(Q_n - K_n)(\tau_j) = 0 \tag{7}$$

and

$$(w(Q_n - K_n))'(\tau_j) = 0 \tag{8}$$

for $j = 1, 2, ..., r$. But by Lemma 3, $n \leqslant r \leqslant n+1$. If $r = n$, then (7) and (8) yield that $w(Q_n - K_n)$ has at least $2n$ zeros. This is impossible unless $Q_n \equiv K_n$, since otherwise $w(Q_n - K_n)$ $(\not\equiv 0)$ has n distinct zeros τ_j $(j = 1, 2, ..., n)$ from (7), which would imply that $(w(Q_n - K_n))'$ has exactly $(n+1)$ zeros and all of them are separated by the τ_j's by Lemma 2. If $r = n+1$, then (7) implies $Q_n \equiv K_n$. ∎

From now on, we denote the unique solution of (P_y) by $Q_n = Q_n(\cdot, y)$, and assume r, λ_j's, and τ_j's are associated with $Q_n(\cdot, y)$ as in (4). Let $T_n(x, w)$ be the weighted Chebyshev polynomial of degree n and denote

$$\hat{T}_n(x) = \hat{T}_n(x, w) = \frac{T_n(x, w)}{\|wT_n(\cdot, w)\|}.$$

LEMMA 5. *If $r = n+1$, then $Q_n = \hat{T}_n$ or $-\hat{T}_n$.*

Proof. Each τ_j is a zero of $(wQ_n)'$ $(j = 1, 2, ..., n+1)$. Thus $(wQ_n)'$ has no other zeros by the second half of Lemma 3. So Rolle's theorem implies that we can not have $(wQ_n)(\tau_j) = (wQ_n)(\tau_{j+1}) (= \pm \|wQ_n\|)$ for any j, so necessarily, wQ_n has alternating signs at the points τ_j, $j = 1, 2, ..., n+1$. Hence $Q_n = \hat{T}_n$ or $-\hat{T}_n$ by the maximal equioscillation property. ∎

LEMMA 6. *If $r = n$, then $\lambda_j \lambda_{j+1} < 0$, $j = 1, 2, ..., n-1$.*

Proof. For $j = 1, 2, ..., n-1$, define

$$p_{j,j+1}(x) := (x-y)^2 \prod_{\substack{k=1 \\ k \neq j, j+1}} (x - \tau_k) \in \mathscr{P}_n.$$

By (4),

$$0 = \lambda_j (wp_{j,j+1})(\tau_j) + \lambda_{j+1} (wp_{j,j+1})(\tau_{j+1}). \tag{9}$$

Note that $\lambda_i \neq 0$, $y \neq \tau_i$, $w(\tau_i) > 0$, $i = 1, 2, ..., n$, and $p_{j,j+1}$ has no sign changes in (τ_{j-1}, τ_{j+2}) $(\tau_{-1} := -\infty$, $\tau_{n+1} := +\infty)$. Now the lemma follows from (9). ∎

LEMMA 7. *There exist α_j, $\beta_j \in [-\infty, +\infty]$, $j = 1, 2, ..., n+2$ with $\alpha_1 = -\infty$, $\beta_{n+2} = +\infty$, and $\alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1}$, $j = 1, 2, ..., n+1$, such that*

$$r(y) = n+1 \text{ if and only if } y \in \bigcup_{j=1}^{n+2} (\alpha_j, \beta_j).$$

Proof. Let us denote the extremal points of $w\hat{T}_n$ by $\hat{\tau}_1, \hat{\tau}_2, ..., \hat{\tau}_{n+1}$ with

$$\hat{\tau}_1 < \hat{\tau}_2 < \cdots < \hat{\tau}_{n+1}.$$

Define the resolvent of $\hat{T}_n$ by (cf. [9])

$$R(x) := \prod_{j=1}^{n+1} (x - \hat{\tau}_j), \tag{10}$$

and set

$$R_k(x) := \frac{R(x)}{(x - \hat{\tau}_k)}, \qquad k = 1, 2, ..., n+1. \tag{11}$$

Assume $i < j$. Note that

$$R_i(x) - R_j(x) = \frac{\hat{\tau}_i - \hat{\tau}_j}{x - \hat{\tau}_i} R_j(x).$$

So at those points x where $(wR_j)'(x) = 0$, we have

$$\frac{(wR_i)'(x)}{(wR_j)(x)} = \frac{\hat{\tau}_j - \hat{\tau}_i}{(x - \hat{\tau}_i)^2} > 0. \tag{12}$$

Since, from Lemma 2, $(wR_k)'$ $(k = 1, 2, ..., n+1)$ has exactly $(n+1)$ distinct zeros, equation (12) implies that the zeros of $(wR_i)'$ and that of $(wR_j)'$ are interlacing. If we denote the zeros of $(wR_k)'$ by

$$\zeta_1^{(k)} < \zeta_2^{(k)} < \cdots < \zeta_{n+1}^{(k)},$$

for $k = 1, 2, ..., n+1$, then

$$-\infty =: \zeta_0^{(1)} < \zeta_1^{n+1} < \cdots < \zeta_1^{(2)} < \zeta_1^{(1)} < \zeta_2^{(n+1)} < \cdots$$
$$< \zeta_n^{(1)} < \zeta_{n+1}^{(n+1)} < \zeta_{n+1}^{(n)} < \cdots < \zeta_{n+1}^{(1)} < \zeta_{n+2}^{(n+1)} := +\infty.$$

From this we claim:

> $(wR_k)'(y)$, $k = 1, 2, ..., n+1$, have the same sign if and only if $y \in \bigcup_{j=0}^{n+1} (\zeta_j^{(1)}, \zeta_{j+1}^{(n+1)})$. (13)

In fact, from Lemma 2 and the fact that $(wR_k)(x) > 0$ (as $x \to +\infty$), we know $(wR_k)(\zeta_{n+1}^{(k)}) > 0$ and $(wR_k)'$ has no zero in $(\zeta_{n+1}^{(k)}, +\infty)$. So $\operatorname{sgn}(wR_k)'(y) = -1$ for $y > \zeta_{n+1}^{(k)}$. Since $(wR_k)'$ only changes its sign at $\zeta_j^{(k)}$ $(j = 1, 2, ..., n+1)$, it then follows that $\operatorname{sgn}(wR_k)'(y) = (-1)^{n-j}$ for all $k = 1, 2, ..., n+1$, if and only if $y \in (\zeta_j^{(1)}, \zeta_{j+1}^{(n+1)})$, $(j = 0, 1, ..., n = 1)$. This proves the claim (13).

Define $\alpha_j := \zeta_{j-1}^{(1)}$, and $\beta_j := \zeta_j^{(n+1)}$, $j = 1, 2, ..., n+2$. If $y \in \bigcup_{j=1}^{n+2} (\alpha_j, \beta_j)$, then by using Lagrange's interpolation formula associated with points $\hat{\tau}_j$, $j = 1, 2, ..., n+1$, we can verify that (4) is satisfied with $r(y) = n+1$, $\tau_j = \hat{\tau}_j$, $Q_n = (\operatorname{sgn}(w\hat{T}_n)'(y))\, \hat{T}_n$ and $\lambda_j = \hat{\lambda}_j$ with

$$\hat{\lambda}_j := \frac{(-1)^{n+1-j}}{w(\hat{\tau}_j)\, |R_j(\hat{\tau}_j)|} (wR_j)'(y) \operatorname{sgn}(w\hat{T}_n)'(y), \qquad j = 1, 2, ..., n+1. \tag{14}$$

Hence $\hat{T}_n$ or $-\hat{T}_n$ is the solution of (P_y).

Conversely, if for $y \in R$ we have $r(y) = n+1$, then by Lemma 5, $Q_n = \hat{T}_n$ or $-\hat{T}_n$. So $\tau_j = \hat{\tau}_j$. In (4), $\operatorname{sgn} \lambda_j = \operatorname{sgn} Q_n(\hat{\tau}_j)$ and Q_n has the equioscillation property, so

$$\lambda_j \lambda_{j+1} < 0, \qquad j = 1, 2, ..., n. \tag{15}$$

Substituting R_j $(j = 1, 2, ..., n+1)$ for p_n in (4), we can solve for λ_j. The formula for λ_j is the same as that for $\hat{\lambda}_j$ in (14). But this time (15) holds, thus $(wR_j)'(y)$, $j = 1, 2, ..., n+1$, must be of the same sign, therefore $y \in \bigcup_{j=1}^{n+2} (\alpha_j, \beta_j)$ by claim (13). ∎

LEMMA 8. *Let α_j's and β_j's be defined as in Lemma 7. We have $Q_n = \hat{T}_n$ or $-\hat{T}_n$ at $y = \alpha_{k+1}$ or β_k, $k = 1, 2, ..., n+1$.*

Proof. According to $y = \alpha_{k+1}$ or $y = \beta_k$, we have $\hat{\lambda}_1 = 0$ or $\hat{\lambda}_{n+1} = 0$ in (14), respectively. In any of the two cases mentioned above, (4) is always satisfied for $Q_n = (\operatorname{sgn}(w\hat{T}_n)'(y))\, \hat{T}_n$ by using $\hat{\lambda}_1, ..., \hat{\lambda}_n$ or $\hat{\lambda}_2, ..., \hat{\lambda}_{n+1}$ of (14) as $\lambda_1, ..., \lambda_n$ in (4). Consequently, $r(y) = n$ and $Q_n = \hat{T}_n$ or $-\hat{T}_n$. ∎

Let $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ be the extremal points of $w\hat{T}_{n-1}$, and

$$R^*(x) := \prod_{j=1}^{n} (x - \sigma_j). \tag{16}$$

Let $\xi_1 < \xi_2 < \cdots < \xi_{n+1}$ be all the zeros of $(wR^*)'$.

LEMMA 9. *For $j = 1, 2, ..., n+1$, there holds*

$$\beta_j < \xi_j < \alpha_{j+1}.$$

Proof. Let $e = e^+$ or e^- where $e^{\pm}(x) := (w\hat{T}_n)(x) \pm (w\hat{T}_{n-1})(x)$, then

$$(-1)^{n+1-j} e(\hat{\tau}_j) \geq 0, \qquad j = 1, 2, ..., n+1,$$

so e has at least one zero in each interval $[\hat{\tau}_j, \hat{\tau}_{j+1}]$, $j = 1, 2, ..., n$. If we count $\hat{\tau}_j$ twice when $e(\hat{\tau}_j) = 0$, then e has at least n zeros. But, from its form, e can have at most n zeros (if we count z twice when $e(z) = 0$ and e does not change its sign at z). So e has exactly n zeros with our method of zero counting. Thus each interval $[\hat{\tau}_j, \hat{\tau}_{j+1}]$ $(j = 1, 2, ..., n)$ contains exactly one point at which e vanishes. We now claim that each interval $[\hat{\tau}_j, \hat{\tau}_{j+1}]$ $(j = 1, 2, ..., n)$ contains exactly one of σ_j's. In fact, if σ_k and σ_{k+1} are both contained in $[\hat{\tau}_j, \hat{\tau}_{j+1}]$, then either e^+ or e^- will have at least two distinct zeros in $[\hat{\tau}_j, \hat{\tau}_{j+1}]$, a contradiction. Hence

$$\hat{\tau}_1 < \sigma_1 < \hat{\tau}_2 < \cdots < \hat{\tau}_n < \sigma_n < \hat{\tau}_{n+1}.$$

So the zeros of wR^* and those of wR_1 (or wR_{n+1}) are interlacing. With the help of Lagrange's interpolation formula, it then follows that the zeros of $(wR^*)'$ and that of $(wR_1)'$ (resp. $(wR_{n+1})'$) are also interlacing. Therefore

$$\xi_j < \zeta_j^{(1)} \text{ (resp. } \zeta_j^{(n+1)} < \xi_j), \qquad j = 1, 2, ..., n+1,$$

which yields the lemma. ▮

LEMMA 10. *We have $Q_n(\cdot, y) = \hat{T}_{n-1}$ or $-\hat{T}_{n-1}$ if and only if $y = \xi_j$ for some $j = 1, 2, ..., n+1$.*

Proof. If $Q_n(\cdot, y) = \pm \hat{T}_{n-1}$, then $\tau_j \in \{\sigma_k; k = 1, 2, ..., n\}$ in (4). Taking $p_n = R^*$ in (4) gives $(wR^*)'(y) = 0$, so $y = \xi_j$ for some j.

Conversely, if $y = \xi_j$, then $(wR^*)'(y) = 0$. For any $p_n \in \mathscr{P}_n$, let a_n be the coefficient of x^n in p_n, then Lagrange's interpolation formula will give us

$$p_n(x) - a_n R^*(x) = \sum_{j=1}^{n} \frac{(wp_n)(\sigma_j)}{w(\sigma_j)\, R^{*\prime}(\sigma_j)} \frac{R^*(x)}{(x - \sigma_j)}.$$

Multiplying both sides of the above equation by $w(x)$ and then differentiating the resulting products with respect to x and then evaluating at $x = y$ will yield

$$(wp_n)'(y) = \sum_{j=1}^{n} \frac{1}{w(\sigma_j)\, R^*(\sigma_j)} \left(-\frac{(wR^*)(y)}{(y - \sigma_j)^2} \right) (wp_n)(\sigma_j).$$

We can see that with $Q_n = (\operatorname{sgn}(w\hat{T}_{n-1})'(y))\, \hat{T}_{n-1}$, (4) will be satisfied if $r = n$, $\tau_j = \sigma_j$ and

$$\lambda_j = \frac{(-1)^{n-j}}{w(\sigma_j)\, |R^{*\prime}(\sigma_j)|} \left(-\frac{(wR^*)(y)}{(y - \sigma_j)^2} \right) \operatorname{sgn}(w\hat{T}_{n-1})'(y), \qquad j = 1, 2, ..., n.$$

This completes the proof of the lemma. ▮

LEMMA 11. *Let $Q_n(x, y) = \sum_{j=0}^{n} a_j(y)\, x^j$ (the unique solution of (P_y)), then $a_j(y)$ is a continuous function of y ($j = 0, 1, ..., n$.)*

Proof. Let $y_0 \in R$ and $y^{(m)} \to y_0$ (as $m \to \infty$). Since

$$\left\| w(x) \sum_{j=0}^{n} a_j(y^{(m)})\, x^j \right\| \leqslant 1,$$

there exists a number $M > 0$ such that, for $j = 0, 1, ..., n$,

$$|a_j(y^{(m)})| \leqslant M, \qquad m = 1, 2, ...$$

Let Λ be any infinite subset of $\{1, 2, 3, ...\}$. Then $\{(a_0(y^{(k)}), ..., a_n(y^{(k)}))\}_{k \in \Lambda}$ will have a limit point, say $(a_0^*, ..., a_n^*)$. Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of Λ such that

$$\lim_{k \to \infty} a_j(y^{(n_k)}) = a_j^*, \qquad j = 0, 1, ..., n.$$

Then

$$\lim_{k \to \infty} \sum_{j=0}^{n} a_j(y^{(n_k)})\, x^j = \sum_{j=0}^{n} a_j^* x^j,$$

locally uniformly in R. Let $Q^*(x) := \sum_{j=0}^{n} a_j^* x^j$. Note that for any $p_n \in \mathscr{P}_n$ with $\|wp_n\| \leqslant 1$,

$$\frac{\partial}{\partial x} (w(x)\, Q_n(x, y^{(n_k)}))|_{x = y^{(n_k)}} \geqslant (wp_n)'\,(y^{(n_k)}).$$

So, by letting $k \to \infty$,

$$(wQ^*)'\,(y_0) \geqslant (wp_n)'\,(y_0).$$

With the notation $\|f\|_{[a,b]} := \sup_{x \in [a,b]} |f(x)|$ and the fact that there exist finite real numbers a_n and b_n such that

$$\|wp_n\|_{[a_n, b_n]} = \|wp_n\|$$

for all $p_n \in \mathscr{P}_n$ (cf. [7]), we find

$$\|wQ^*\| = \|wQ^*\|_{[a_n, b_n]} = \lim_{k \to \infty} \|w(x)\, Q_n(x, y^{(n_k)})\|_{[a_n, b_n]} = 1.$$

Hence Q^* is a solution of (P_{y_0}). By the uniqueness, $Q^*(x) = Q_n(x, y_0)$ and it follows that the limit $\lim_{y \to y_0} a_j(y)$ exists and equals $a_j(y_0)$ $(j = 0, 1, ..., n)$. ∎

3. Proof of Theorem 1

Let $N_n(y)$, $R(x)$, and $R_k(x)$ have the same meaning as in Section 2 (cf. formulas (3), (10), and (11)). Note that

$$(wR_k)'\,(x) = -\frac{(wR)(x)}{(x - \hat{t}_k)^2} \qquad \text{if} \quad (wR)'\,(x) = 0.$$

Thus, if we use $\theta_0 < \theta_1 < \cdots < \theta_{n+1}$ to denote all the zeros of $(wR)'$, then the zeros of $(wR)'$ and that of $(wR_1)'$ (or $(wR_{n+1})'$) are interlacing, so

$$\alpha_{j+1} < \theta_j < \beta_{j+1}, \qquad j = 0, 1, ..., n+1,$$

where $\alpha_{j+1} = \zeta_j^{(1)}$ and $\beta_{j+1} = \zeta_{j+1}^{(n+1)}$, $j = 0, 1, ..., n+1$, are defined as in Lemma 7.

Similarly, we can also verify the following:

$$\beta_j < \hat{\tau}_j < \alpha_{j+1}, \qquad j = 1, 2, ..., n+1. \tag{17}$$

From this point on, we shall take $w(x) = w_2(x) = e^{-x^2}$.

We need the following additional lemmas for the proof of Theorem 1.

LEMMA 12. *For $j = 1, 2, ..., n+2$, in each interval (α_j, β_j), function $N_n(y)$ has a unique local maximum point at θ_{j-1}.*

Proof. Let $y \in (\alpha_j, \beta_j)$. From Lemma 7 and its proof, $Q_n = (\operatorname{sgn}(w_2 \hat{T}_n)'(y)) \hat{T}_n$. So,

$$-\frac{\partial}{\partial x}(w_2(x) Q_n(x, y))|_{x=y} = \frac{2 \operatorname{sgn}(w_2(\hat{T}_n)'(y))}{\|w_2 T_n(\cdot, w_2)\|}(w_2 R)(y),$$

by using the definition of R and comparing the leading coefficients on both sides. But the left hand side equals $-N_n(y) < 0$ (cf. the remark after the proof of Lemma 3). So the right hand side does not change sign for $y \in (\alpha_j, \beta_j)$. Thus $(w_2 \hat{T}_n)'(y)$ has the same sign for $y \in (\alpha_j, \beta_j)$, so $Q_n(x, y) \equiv \hat{T}_n(x)$ or $Q_n(x, y) \equiv -\hat{T}_n(x)$ for such y (Q_n is independent of y in this case!). Assume $Q_n \equiv \hat{T}_n$. Then

$$N_n(y) = (w_2 \hat{T}_n)'(y) = -\frac{2(w_2 R)(y)}{\|w_2 T_n(\cdot, w_2)\|} > 0. \tag{18}$$

So $w_2 R(y) < 0$ for $y \in (\alpha_j, \beta_j)$. It then follows easily that $w_2 R$ has a unique minimum point at θ_{j-1}, so $N_n(y)$ has a unique maximum point at θ_{j-1}. The case when $Q_n \equiv -\hat{T}_n$ can be handled similarly. ∎

Next, we examine the behavior of $N_n(y)$ when $y \in (\beta_j, \alpha_{j+1})$, $j = 1, 2, ..., n+1$. Recall that ξ_j's are all the zeros of $(w_2 R^*)'$ with R^* defined in (16).

LEMMA 13. *The function $N_n(y)$ is decreasing in (β_j, ξ_j) and increasing in (ξ_j, α_{j+1}), $j = 1, 2, ..., n+1$.*

Proof. Let $y \in (\beta_j, \alpha_{j+1})$. By Lemmas 5, 6, and 7, $r(y) = n$, $\lambda_i \lambda_{i+1} < 0$ $(i = 1, 2, ..., n-1)$ in (4), and

$$n - 1 \leqslant \partial Q_n := \text{the degree of } Q_n(\cdot, y) \leqslant n.$$

If $\partial Q_n = n-1$, then $Q_n = \hat{T}_{n-1}$ or $-\hat{T}_{n-1}$ by the maximal equioscillation property; and this happens only when $y = \xi_j$ by Lemma 10.

Now assume $\partial Q_n = n$, then there exist

$$t_1 < t_2 < \cdots < t_n \qquad \text{and} \qquad t$$

(the zeros of $(w_2 Q_n)'$) such that the following relations hold:

$$(w_2 Q_n)(t_i) = (-1)^i \varepsilon \ (\varepsilon = \pm 1), \qquad i = 1, 2, ..., n$$

and

$$\frac{\partial}{\partial x}(w_2(x)\, Q_n(x, y))|_{x=t} = 0.$$

Set $\hat{R}(x) := \prod_{i=1}^{n} (x - t_i)$, and let a_n be the leading coefficient of Q_n. Then $a_n \neq 0$ and

$$\frac{\partial}{\partial x}(w_2(x)\, Q_n(x, y)) = -2a_n w_2(x)\, \hat{R}(x)(x - t).$$

So

$$(w_2 \hat{R})'(x) = -\frac{(\partial^2/\partial x^2)(w_2(x)\, Q_n(x, y))}{2a_n(x-t)} + \frac{(\partial/\partial x)(w_2(x)\, Q_n(x, y))}{2a_n(x-t)^2}.$$

But by (4), $(w_2 \hat{R})'(y) = 0$ since $\hat{R}(t_i) = 0$, $i = 1, 2, ..., n$, thus

$$\frac{\partial^2}{\partial x^2}(w_2 Q_n(x, y))|_{x=y} = \frac{(\partial/\partial x)(w_2(x)\, Q_n(x, y))|_{x=y}}{y - t} \neq 0, \tag{19}$$

for $y \in (\beta_j, \alpha_{j+1}) \backslash \{\xi_j\}$, $j = 1, 2, ..., n+1$. Therefore, y is not a local extremum point of $(\partial/\partial x)(w_2(x)\, Q_n(x, y))$ as a function of x.

By Lemma 8, $Q_n(x, \beta_j) = \text{sgn}(w_2 \hat{T}_n)'(\beta_j)\, \hat{T}_n(x)$, so

$$\begin{aligned} N_n(\beta_j) &= \text{sgn}(w_2 \hat{T}_n)'(\beta_j)(w_2 \hat{T}_n)'(\beta) \\ &= -\frac{2\, \text{sgn}(w_2 \hat{T}_n)'(\beta_j)(w_2 R)(\beta_j)}{\|w_2 T_n(\cdot, w_2)\|} > 0, \end{aligned}$$

where the second equality depends on (18). Now by (17) and the fact the $w_2 R$ is monotone between θ_{j-1} and θ_j, we conclude that

$$(w_2 R)(\beta_j)(w_2 R)'(\beta_j) < 0,$$

so

$$\operatorname{sgn}(w_2 \hat{T}_n)'(\beta_j)(w_2 R)'(\beta_j) > 0.$$

Hence

$$-\frac{\partial^2}{\partial x^2}(w_2(x)\, Q_n(x, \beta_j))|_{x=\beta_j}$$
$$= \frac{2 \operatorname{sgn}(w_2 \hat{T}_n)'(\beta_j)(w_2 R)'(\beta_j)}{\|w_2 T_n(\cdot, w_2)\|} > 0.$$

Now, by Lemma 11, for $y > \beta_j$ and y close enough to β_j,

$$-\frac{\partial^2}{\partial x^2}(w_2(x)\, Q_n(x, y))|_{x=y} > 0. \tag{20}$$

But in view of (19), this implies that (20) holds for all $y \in (\beta_j, \xi_j)$. Now the continuity of $-(\partial^2/\partial x^2)(w_2(x)\, Q_n(x, y))$ in (x, y) implies that, for every $y \in (\beta_j, \xi_j)$, there is $\delta := \delta(y)$ such that

$$-\frac{\partial^2}{\partial x^2}(w_2(x)\, Q_n(x, y^*))|_{x=x^*} > 0$$

whenever $|x^* - y| < \delta$ and $|y^* - y| < \delta$. Thus, if $y_1 < y_2$ and $y_1, y_2 \in (y-\delta, y+\delta)$,

$$N_n(y_2) = \frac{\partial}{\partial x}(w_2(x)\, Q_n(x, y_2))|_{x=y_2}$$
$$< \frac{\partial}{\partial x}(w_2(x)\, Q_n(x, y_2))|_{x=y_1}$$
$$\leqslant N_n(y_1).$$

Hence N_n is decreasing in $(y-\delta, y+\delta)$. Consequently, N_n is decreasing in (β_j, ξ_j), $j = 1, 2, ..., n+1$.

Similarly, we can show that N_n is increasing in (ξ_j, β_{j+1}), $j = 1, 2, ..., n+1$. ∎

Proof of Theorem 1. Combining Lemmas 12 and 13, we have

$$N_n(y) < \|(w_2 \hat{T}_n)'\| \qquad \text{for} \quad x \in \bigcup_{j=1}^{n+2} (\alpha_j, \beta_j).$$

Together with Lemmas 5 and 7, it then follows that

$$\|(w_2 p_n)'\| < \|(w_2 \hat{T}_n)'\| \; \|w_2 p_n\|,$$

for all $p_n \in \mathscr{P}_n$ unless

$$\frac{p_n}{\|w_2 p_n\|} = \pm \hat{T}_n.$$

This completes our proof of Theorem 1. ∎

4. Remarks

(I) By a linear transformation, one can easily prove

COROLLARY 14. *For real numbers $a > 0$, b, and c, equality* (1) *is true for the weight*

$$w_H(x) = e^{\,(ax^2 + bx + c)}.$$

(II) From [1], we know that

$$An^{1/2} \leqslant C_{w_2}(n) \leqslant Bn^{1/2},$$

where $A > 0$ and $B > 0$ are absolute constants. So using Theorem 1 we get the following estimate of $(w_2 \hat{T}_n)'$:

COROLLARY 15. *There are absolute constants $A > 0$ and $B > 0$ such that*

$$An^{1/2} \leqslant \|(w_2 \hat{T}_n)'\| \leqslant Bn^{1/2}.$$

The asymptotics of $\hat{T}_n(z, w_2)$ in $C \backslash [-1, 1]$ is obtained in [5] (more general weights are considered there). Hence, in view of Corollary 15, it is desirable to ask: what is the asymptotics of $\|(w_2 \hat{T}_n)'\|$?

(III) If $(w_2 p_n)'$ is replaced by $w_2 p_n'$ in (2), then it is not clear if Chebyshev polynomial $T_n(\cdot, w_2)$ will still give us the best constant. Our method of proof of Theorem 1 cannot be directly applied to solve this problem.

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